The Fujita phenomenon in exterior domains under the Robin boundary conditions

Jean-François Rault

LMPA Joseph Liouville, ULCO, FR 2956 CNRS,
Université Lille Nord de France
50 rue F. Buisson, B.P. 699, F-62228 Calais Cedex (France)
jfrault@lmpa.univ-littoral.fr

Abstract

The Fujita phenomenon for nonlinear parabolic problems $\partial_t u = \Delta u + u^p$ in an exterior domain of $\mathbb{R}^N$ under the Robin boundary conditions is investigated in the superlinear case. As in the case of Dirichlet boundary conditions (see Trans. Amer. Math. Soc 316 (1989), 595-622 and Isr. J. Math. 98 (1997), 141-156), it turns out that there exists a critical exponent $p = 1+2/N$ such that blow-up of positive solutions always occurs for subcritical exponents, whereas in the supercritical case global existence can occur for small non-negative initial data.

Key words: Nonlinear parabolic problems; Robin boundary conditions ; Global solutions; Blow-up.

1 Introduction

Let Ω be an exterior domain of $\mathbb{R}^N$, that is to say a connected open set Ω such that $\overline{\Omega}^c$ is a bounded domain when $N \geq 2$, and in dimension one, Ω is the complement of a real closed interval. We always suppose that the boundary $\partial\Omega$ is of class $\mathcal{C}^2$. The outer normal unit vector field is denoted by $\nu : \partial\Omega \to \mathbb{R}^N$ and the outer normal derivative by ∂_ν. Let p be a real number with $p > 1$, α a non-negative continuous function on $\partial\Omega \times \mathbb{R}^+$ and φ a continuous function in $\overline{\Omega}$. Consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases} \tag{1}$$

In this paper, we give a positive answer to Levine & Zhang's question [1]: the Fujita phenomenon, well-known in the case of $\Omega = \mathbb{R}^N$ (see Ref. [2]), remains true for the Robin boundary conditions. The case limiting $\alpha \equiv 0$ and $\alpha = +\infty$

were proved by Levine & Zhang in [1] and by Bandle & Levine in [3], respectively. The real number $1+\frac{2}{N}$ is still the critical exponent, and we prove the blowing-up of all positive solutions of Problem (1) for subcritical exponents p, whereas in the supercritical case, we show the existence of global positive solutions of Problem (1) for sufficiently small initial data. In the last section, we study the case of a general second order elliptic operator replacing the Laplacian. We also consider a non-linearity including a time and a space dependence. Throughout, we shall assume that α is non-negative

$$\alpha \geq 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \tag{2}$$

and, in order to deal with classical solutions, we need some regularity on α

$$\alpha \in \mathcal{C}(\partial\Omega \times \mathbb{R}^+). \tag{3}$$

To construct solutions with the truncation procedure (see Section 2), we suppose

$$\varphi \in \mathcal{C}(\overline{\Omega}),\ 0 <\parallel \varphi \parallel_\infty < \infty,\ \varphi \geq 0,\ \lim_{\|x\|_2 \to \infty} \varphi(x) = 0. \tag{4}$$

In the case $\Omega = \mathbb{R}^N$, the boundary conditions are dropped, and the result is well-known by the classical paper of Fujita [2]. Thus we suppose $\Omega \neq \mathbb{R}^N$.

2 Preliminaries

First, we give the definition of positive solution which is understood along this paper.

Definition 2.1 *A positive solution of Problem (1) is a positive function $u : (x,t) \mapsto u(x,t)$ of class $\mathcal{C}(\overline{\Omega} \times [0,T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$, satisfying*

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0,+\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

where α and φ are given with (2), (3) and (4). The time $T = T(\alpha,\varphi) \in (0,+\infty]$ denotes the maximal existence time of the solution u. If $T = +\infty$, the solution is called global.

From [3], if $T < +\infty$, u blows up in finite time, that is to say:

$$\lim_{t \nearrow T} \sup_{x \in \overline{\Omega}} u(x,t) = +\infty.$$

Then, let us recall a standard procedure to construct solutions of Problem (1) in outer domains for uniformly bounded and continuous initial data φ. For more details, we refer to [4] , [5] and references therein. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of nested bounded domains such that

$$\overline{\Omega}^c \subseteq D_0 \subseteq D_1 \subseteq \cdots \subseteq \bigcup_{n\in\mathbb{N}} D_n = \mathbb{R}^N.$$

Let u_n be the solution of

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \cap D_n \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial D_n \times (0, +\infty), \\ u(\cdot, 0) = \varphi_n & \text{in } \overline{\Omega} \cap D_n, \end{cases} \tag{5}$$

where $(\varphi_n)_{n\in\mathbb{N}}$ denotes a sequence of functions in $\mathcal{C}_0(\overline{\Omega} \cap D_n)$ such that

$$0 \leq \varphi_n \leq \varphi \text{ in } \overline{\Omega} \cap D_n$$

and $\varphi_n \to \varphi$ uniformly in any compact of $\overline{\Omega} \cap D_n$ as $n \to +\infty$. Let z denote the solution of the ODE

$$\begin{cases} \dot{z} = z^p, \\ z(0) = \| \varphi \|_\infty, \end{cases}$$

with maximal existence time $S = \frac{1}{(p-1)\|\varphi\|_\infty^{p-1}}$. By the comparison principle (see [6]), we have

$$0 \leq u_n(x,t) \leq u_{n+1}(x,t) \leq z(x,t) \text{ in } \overline{\Omega} \cap D_n \times [0, S].$$

Standard arguments based on a priori estimates for the heat equation imply $u_n \to u$ in the sense of $\mathcal{C}^{2,1}_{loc}(\overline{\Omega} \times (0, S))$ as $n \to +\infty$, where u is a positive solution of Problem (1). Moreover, since u_n vanishes on ∂D_n for each $n \in \mathbb{N}^*$, the solution u vanishes at infinity:

$$\lim_{\|x\|_2 \to \infty} u(x,t) = 0 \ , \forall\ t \in (0, T).$$

3 Blow up case

In this section, we compare the solution of Problem (1) with an appropriate Dirichlet solution. We prove the following theorem:

Theorem 3.1 *Suppose that conditions (2), (3) and (4) are fullfiled. Then all non-trivial positive solutions of Problem (1) blow up in finite time for $p \in (1, 1+2/N)$. Moreover, if $N \geq 3$, blow up also occurs for $p = 1 + 2/N$.*

Proof: Ab absurdo, suppose that there exists α and a non-trivial φ satisfying the hypotheses above, and such that the solution u of Problem (1) with these parameters is global. Then, consider u_n the solution of the truncated Problem (5). By the comparison principle from [6], we obtain

$$0 \leq u_n(x,t) \leq u(x,t) \text{ in } \overline{\Omega} \cap D_n \text{ for } t > 0.$$

Thus, u_n can not blow up in finite time, and u_n must be global. Next, define v_n the solution of the following problem

$$\begin{cases} \partial_t v_n = \Delta v_n + v_n^p & \text{in } \overline{\Omega} \cap D_n \times (0, +\infty), \\ v_n = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v_n = 0 & \text{on } \partial D_n \times (0, +\infty), \\ v_n(\cdot, 0) = \varphi_n & \text{in } \overline{\Omega} \cap D_n. \end{cases}$$

Again, the comparison principle from [6] implies $0 \leq v_n(x,t) \leq u_n(x,t)$ in $\overline{\Omega} \cap D_n$ for $t > 0$. Then, we consider v the solution of the Dirichlet problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

obtained as the limit of the v_n by the truncation procedure described in Section 2. Thus, $v \leq u$ in $\overline{\Omega} \times (0, +\infty)$ and v is a global positive solution. A contradiction with Bandle & Levine results [3] (see [7] for the one-dimensional case). If $N \geq 3$ and $p = 1 + 2/N$, the contradiction holds with Mochizuki & Suzuki's results [8] and [9]. Hence, our solution u must blow up in finite time.

■

4 Global existence case

Now, we consider supercritical exponents:

$$p > 1 + \frac{2}{N}.$$

We look for a global positive super-solution of Problem (1), we mean a function U satisfying

$$\begin{cases} \partial_t u \geq \Delta u + u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u \geq 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) \geq \varphi & \text{in } \overline{\Omega}. \end{cases}$$

With this global super-solution and using the comparison principle, we construct the sequence $(u_n)_{n \in \mathbb{N}}$ of global positive solutions of Problems (5). Thus, using the truncation procedure of Section 2, we construct a global positive solution of Problem (1). We use two different super-solutions, and we obtain two results on the global existence with some restrictions on the dimension N or on the coefficient α. First, we only suppose that the dimension

$$N \geq 3.$$

Theorem 4.1 *Under hypotheses (2), (3) and (4), for $N \geq 3$ and*

$$p > 1 + \frac{2}{N},$$

Problem (1) admits global non-trivial positive solutions for sufficiently small initial data φ.

Proof: Consider φ satisfying (4) and v the non-trivial positive solution v of the Neumann problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

where the initial data φ is sufficiently small such that the solution v is global. This choice can be achieved because $N \geq 3$ and $p > 1 + 2/N$, see Levine & Zhang [1]. For all $\alpha \geq 0$ on $\partial\Omega \times (0, +\infty)$, we obtain

$$\partial_\nu v + \alpha v \geq 0 \text{ on } \partial\Omega \times (0, +\infty).$$

Thus, v is a super-solution of Problem (1), and we can deduce the statement of the theorem.

■

Now, we suppose that there exists a positive constant $c > 0$ such that

$$\alpha \geq c \text{ on } \partial\Omega \times \mathbb{R}^+. \tag{6}$$

We do not impose any condition on the dimension.

Theorem 4.2 *Let α be a coefficient satisfying (3) and (6), φ an initial data with (4). For*

$$p > 1 + \frac{2}{N},$$

Problem (1) admits global positive solutions for sufficiently small initial data φ.

Proof: We consider the function $U : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$U(x,t) = A(t+t_0)^{-\mu} \exp\Big(- \frac{\| x \|_2^2}{4(t+t_0)}\Big),$$

where $\mu = 1/(p-1)$, $t_0 > 0$ and $A > 0$ will be chosen below. All the calculus will be detailed in the proof of the general Theorem 5.3. If $A > 0$ is small enough, we have

$$\partial_t U \geq \Delta U + U^p \text{ in } \overline{\Omega} \times (0, +\infty).$$

On the boundary $\partial\Omega$, hypothesis (6) gives

$$\begin{aligned} \partial_\nu U(x,t) + \alpha U(x,t) &\geq \Big(\frac{-x\cdot\nu(x)}{2(t+t_0)} + \alpha(x,t)\Big) U(x,t) \\ &\geq \Big(\frac{-x\cdot\nu(x)}{2(t+t_0)} + c\Big) U(x,t) \end{aligned}$$

Since the boundary $\partial\Omega$ is compact, the function $(\partial\Omega \ni x \mapsto -x\cdot\nu(x) \in \mathbb{R}$ is bounded. We choose t_0 sufficiently big such that $-x\cdot\nu(x)/(2t_0) + c \geq 0$. Then we obtain

$$\partial_\nu U + \alpha U \geq 0 \text{ on } \partial\Omega \times (0, +\infty).$$

Finally, if we choose $\varphi \leq U(\cdot, 0)$ in $\overline{\Omega}$, the function U is a super-solution of Problem (1).

■

Remark 4.3 *In the previous proof, one can note that the hypothesis (6) can be relaxed into*

$$\alpha(x,t) \geq \frac{x\cdot\nu(x)}{2(t+t_0)} \textit{ for all } (x,t) \in \partial\Omega \times (0, +\infty). \tag{7}$$

This condition gives us an optimal bound on α only if we know the geometry of the domain Ω. For instance, if

$$\Omega = \{\| x \|_2 > R\},$$

we obtain

$$x \cdot \nu(x) = -R \text{ for all } x \in \partial\Omega.$$

Then, the equation (7) is equivalent to

$$\alpha(x,t) \geq \frac{-R}{2(t+t_0)} \text{ for all } (x,t) \in \partial\Omega \times (0,+\infty).$$

In particular, the previous theorem holds for all non-negative α.

In the one-dimensional case, using symmetry and translation, we can suppose that $\Omega = (-\infty,-1) \cup (1,+\infty)$. Then, without any additional hypothesis on the parameters of Problem (1), we obtain:

Theorem 4.4 *Assume the conditions (2), (3) and (4). For dimension $N = 1$ and*

$$p > 3,$$

Problem (1) admits global positive solutions for sufficiently small initial data φ.

5 Generalization

In the manner of Bandle & Levine's results [7], we generalize our results. We consider the following problem

$$\begin{cases} \partial_t u = Lu + t^q \| x \|_2^s u^p & \text{in } \overline{\Omega} \times (0,+\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \tag{8}$$

where q and s are two positive real numbers, $p > 1$ is a real number, and L stands for the second order elliptic operator

$$L = \sum_{i,j=1}^{N} \partial_{x_i}\Big(a_{ij}(x)\partial_{x_j}\Big) + \sum_{i=1}^{N} b_i(x)\partial_{x_i}.$$

To deal with classical solutions, the coefficients are assumed to be in $\mathcal{C}^2(\overline{\Omega})$. We keep the hypotheses (2), (3) and (4) on the parameters α and φ. In order to state our principal results, we shall introduce some notations.

$$\rho(x) = \sum_{i,j=1}^{N} a_{ij}(x)\frac{x_i x_j}{\| x \|_2^2}.$$

Throughout, we assume that the matrix $A = (a_{ij})_{1\leq i,j\leq N}$ is normalized, so that for some $\nu_0 \in (0,1]$

$$0 < \nu_0 \leq \rho \leq 1 \text{ in } \overline{\Omega}.$$

Denote $b = (b_1, \dots, b_N)$ and let

$$l(x) = \sum_{i,j=1}^{N} \Big(\partial_{x_j} a_{ij}(x) - b_i(x) \Big) x_i,$$

$$l^*(x) = \sum_{i,j=1}^{N} \Big(\partial_{x_j} a_{ij}(x) + b_i(x) \Big) x_i.$$

We can state the following theorem concerning the blow-up case.

Theorem 5.1 *Assume that* $N \geq 2$*,*

$$\operatorname{div} b(x) \leq 0 \ \ in \ \overline{\Omega},$$

and

$$\rho(x) \leq \frac{\operatorname{trace} A(x) + l(x)}{2} \ \ in \ \overline{\Omega}. \tag{9}$$

Then, all non-trivial positive solutions of Problem (8) blow up in finite time for

$$1 < p < 1 + \frac{2 + 2q + s}{N}.$$

Proof: Ab absurdo, we suppose that there exists a non-trivial positive solution v of Problem (8). As in the proof of Theorem 3.1, we deduce that there exists a non-trivial positive solution u of the following Dirichlet problem

$$\begin{cases} \partial_t u = Lu + t^q \parallel x \parallel_2^s u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases}$$

According to Bandle & Levine's results from [7], the solution u blows up in finite time under the above hypotheses. Thus, v must blow up too.

■

For the one-dimensional case, Bandle & Levine weaken the hypothesis (9). Then, we obtain:

Theorem 5.2 *Assume that* $N = 1$*,*

$$\operatorname{div} b(x) \leq 0 \ \ in \ \overline{\Omega},$$

and

$$\Big(\frac{2 + 2q + s}{p - 1} - 2 \Big) a_{11} + l > 0 \ \ in \ \overline{\Omega}.$$

If $1 < p < 3 + 2q + s$*, then all non-trivial positive solutions of Problem (8) blow up in finite time.*

Now, we consider the global existence case.

Theorem 5.3 *Assume that condition (6) is satisfied,*

$$\rho(x) \leq 1 \ \ in \ \overline{\Omega},$$

and

$$2\gamma_0 := \inf_{\overline{\Omega}} \Big(\operatorname{trace} A + l^* \Big) > 0.$$

Then, for any

$$p > 1 + \frac{2 + 2q + s}{2\gamma_0},$$

Problem (8) admits global non-trivial positive solutions if the initial data φ is sufficiently small.

Proof: We consider the function $U : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$U(x,t) = A(t+t_0)^{-\mu} \exp\Big(- \frac{\| x \|_2^2}{4(t+t_0)} \Big),$$

where $\mu = (2 + 2q + s)/(2p-2)$, $t_0 > 0$ and $A > 0$ will be chosen below. We have

$$\partial_t U(x,t) = \Big(\frac{-\mu}{t+t_0} + \frac{\| x \|_2^2}{4(t+t_0)^2} \Big) U(x,t),$$

$$LU(x,t) = \Big(\rho(x) \frac{\| x \|_2^2}{4(t+t_0)^2} - \frac{\operatorname{trace} A + l^*}{2(t+t_0)} \Big) U(x,t),$$

and

$$\partial_\nu U(x,t) = \Big(\frac{-x \cdot \nu(x)}{2(t+t_0)} \Big) U(x,t).$$

On the boundary $\partial\Omega$, we obtain:

$$\partial_\nu U(x,t) + \alpha U(x,t) = \Big(\frac{-x \cdot \nu(x)}{2(t+t_0)} + \alpha \Big) U(x,t).$$

Thanks to hypothesis (6), and because the boundary $\partial\Omega$ is compact, we can choose t_0 sufficiently big such that

$$\frac{-x \cdot \nu(x)}{2t_0} + c \geq 0 \text{ on } \partial\Omega.$$

Thus, $\partial_\nu U(x,t) + \alpha U(x,t) \geq 0$ is achieved on $\partial\Omega \times (0, +\infty)$. Then, in $\overline{\Omega}$, we have

$$\partial_t U(x,t) - LU(x,t) = \Big((1 - \rho(x)) \frac{\| x \|_2^2}{4(t+t_0)^2} + \frac{\operatorname{trace} A + l^* - 2\mu}{2(t+t_0)} \Big) U(x,t).$$

With $\rho \leq 1$, we ignore the t-quadratic term, and by definition of γ_0, we obtain

$$\partial_t U(x,t) - LU(x,t) \geq \Big(\frac{\gamma_0 - \mu}{t+t_0} \Big) U(x,t), \tag{10}$$

with $\gamma_0 - \mu > 0$. On the other hand, $t < t + t_0$ implies

$$t^q \| x \|_2^s U^p(x,t) \leq A^{p-1} \| x \|_2^s (t+t_0)^{q - \mu(p-1)} \exp\Big(\frac{-(p-1) \| x \|_2^2}{4(t+t_0)} \Big) U(x,t).$$

Using the overestimation

$$\Big(\frac{2s}{p-1} \Big)^{\frac{s}{2}} \exp\Big(\frac{-s}{2} \Big) (t+t_0)^{\frac{s}{2}} \geq \| x \|_2^s \exp\Big(- \frac{\| x \|_2^2 (p-1)}{4(t+t_0)} \Big),$$

we obtain

$$t^q \parallel x \parallel_2^s U^p(x,t) \leq A^{p-1}\Big(\frac{2s}{p-1}\Big)^{\frac{s}{2}} \exp\Big(\frac{-s}{2}\Big)(t+t_0)^{\frac{s}{2}+q-\mu(p-1)}U(x,t). \quad (11)$$

By definition of μ, we have $s/2 + q - \mu(p-1) = -1$. Thus, we just have to choose A sufficiently small, equations (10) and (11) give

$$\partial_t U(x,t) - LU(x,t) \geq t^q \parallel x \parallel_2^s U^p(x,t).$$

Finally, if the initial data $\varphi \leq U(\cdot, 0)$ in $\overline{\Omega}$, U is a super-solution of Problem (8), and we can deduce the existence of a solution using the truncation procedure of Section 2.

■

References

[1] H.A. Levine and Q.S. Zhang, The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values, Proc. of the R. Soc. of Edinb. 130A (200) 591-602.

[2] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. of the Fac. of Sci. of the Univ. of Tokyo 13 (1966) 109-124.

[3] C.Bandle and H.A. Levine, On the existence and the nonexistence of global solutions of reaction-diffusion equations in sectorial domains, Trans. Am. Math. Soc. 316 (1989) 595-622.

[4] C. Bandle, J. von Below and W. Reichel, Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, Rend. Lincei Math. Appl. 17 (2006) 35-67.

[5] J-F. Rault, The Fujita phenomenon in exterior domains under dynamical boundary conditions, Asymptot. Anal. 66 (2010) 1-8 .

[6] J. von Below and C. De Coster, A Qualitative Theory for Parabolic Problems under Dynamical Boundary Conditions, J. of Inequal. and Appl. 5 (2000) 467-486.

[7] C. Bandle and H.A. Levine, Fujita type results for convective-like reaction diffusion equations in exterior domains, ZAMP 40 (1989) 665-676.

[8] K. Mochizuki and R. Suzuki, Critical exponent and critical blow up for quasilinear parabolic equations, Isr. J. Math. 98 (1997) 141-156.

[9] R. Suzuki, Critical blow-up for quasilinear parabolic equations in exterior domains, Tokyo J. of Math. 19 (1996) 397-409.